Nonsingular Complete Curves

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Definition

Let L/K be a field extension. L is of transcendence degree n over K if:

- $\exists x_1, \ldots, x_n \in L$ s.t. $L/K(x_1, \ldots, x_n)$ is finite, and
- $K(x_1, \ldots, x_n)$ is isomorphic as a K-algebra to the field of rational functions in n variables over K.

Remark

Note that L/K is an algebraic extension $\iff$ L is of transcendental degree 0 over K.

Remark

We will be interested in transcendence degree 1 extensions. Note that L is of transcendence degree 1 over K iff

- *L/K is not an algebraic extension, and*
- *for every transcendental element $\alpha \in L$, the extension $L/K(\alpha)$ is finite.*

Definition

Let L/K be a field extension. A valuation $v : L^{\times} \to \mathbb{Z}$ is trivial on K if $v(K^{\times}) = \{0\}$. We let $\mathcal{V}(L/K)$ be the set of all surjective valuation of L that are trivial on K.

Definition

Let K be a field. A nonsingular complete curve X over K (sometimes denoted X/K) is a pair $(X, K(X)/K)$ consisting of

- a field extension $K(X)/K$ of transcendence degree 1 over K, and
- a set X identified with the set $\mathcal{V}(K(X)/K)$ through a given bijection between X and $\mathcal{V}(K(X)/K)$.

Definition

Let X/K be a nonsingular curve.

- An element $P \in X$ is called a point.
- The field $K(X)$ is called the field of rational functions on X.

Definition

Let X/K be a nonsingular curve. By definition, to a point $P \in X$ corresponds a valuation $v_P \in \mathcal{V}(K(X)/K)$. We proved that to v_P corresponds a local PID $\mathcal{O}_{v_P}$ which we denote by $\mathcal{O}_P$. The maximal ideal of $\mathcal{O}_P$ is denoted by $\mathcal{M}_P$.

- The ring $\mathcal{O}_P$ is called the ring of rational functions defined at P.
- An element in $\mathcal{O}_P$ is called a function on X defined at P.
- A function $\alpha \in \mathcal{O}_P$ is said to have a zero at P if $\alpha \in \mathcal{M}_P$.

Definition

- For $\alpha \in \mathcal{M}_P$, the integer $v_P(\alpha)$ is called the order of vanishing of α at P.
- A function $\alpha \in K(X) \setminus \mathcal{O}_P$ is said to have a pole at P.
- For such α, the integer $-v_P(\alpha)$ is called the order of the pole of α at P.

Definition

The domain of $\alpha \in K(X)$, denoted by $\mathrm{Dom}(\alpha)$ is the set of points in X where α is defined. That is

$$\mathrm{Dom}(\alpha) = \{P \in X \mid v_P(\alpha) \geq 0\}.$$

For $U \subseteq X$ we let

$$\mathcal{O}_X(U) = \bigcap_{P \in U} \mathcal{O}_P$$

be the ring of functions on X defined everywhere on U.

Definition

Let X/K be a nonsingular complete curve. A subset $U \subseteq X$ is closed if either U is finite or $U = X$. A subset $V \subseteq X$ is open if $X \setminus V$ is closed.

Remark

- *$\emptyset, X$ are both open and closed.*
- *The intersection of every two open sets is open.*
- *The union of any family of open sets is open.*
- *The family of open sets of X is called the Zariski topology of the affine curve X.*

Definition

Let X/K be a nonsingular complete curve. An open set $U \subset X$ is called affine if

- $\mathcal{O}_X(U)$ is a f.g. K-algebra, and
- $\mathcal{O}_X(U)$ is a Dedekind domain, and
- the map

$$U \to \mathrm{Max}(\mathcal{O}_X(U))$$
$$P \mapsto \mathcal{M}_P \cap \mathcal{O}_X(U)$$

  is well-defined and bijective.

Note that an open set U of X is affine if it is in bijection as above with the nonsingular affine curve $(\mathrm{Max}(\mathcal{O}_X(U)), \mathcal{O}_X(U))$.

Definition

Let K be a field. A projective line over K is a nonsingular complete curve, usually denoted by $\mathbb{P}^1/K$ such that the field of functions $K(\mathbb{P}^1) \cong K(x)$ as a K-algebra.

It is customary to denote by ∞ the point of $\mathbb{P}^1/K$ that corresponds to the valuation v_∞.

Claim

Let K be a field. Let $\mathbb{P}^1/K$ be the projective line associated to $K(x)/K$. Then,

$$\mathbb{P}^1 = \left\{ v_{f(x)} \mid f(x) \in K[x] \text{ irreducible and monic } \right\} \cup \left\{ v_\infty \right\}.$$

In particular, when K is algebraically closed, $\mathbb{P}^1 \rightleftarrows K \cup \{\infty\}$.

Claim

Let X/K be a nonsingular complete curve associated to $K(X)/K$. Let $\alpha \in X$. Then,

$$X = \mathrm{Dom}(\alpha) \cup \mathrm{Dom}(1/\alpha).$$

Proof.

Let $P \in X$. Recall that $\mathcal{O}_P$ is a local PID. Thus, as proved in the recitation, at least one of $\alpha, 1/\alpha$ is in $\mathcal{O}_P$. By definition then, $P \in \mathrm{Dom}(\alpha) \cup \mathrm{Dom}(1/\alpha)$. □

Theorem

Let X/K be a nonsingular complete curve associated to $K(X)/K$. Let $\alpha \in K(X)$ be such that $K(X)/K(\alpha)$ is a finite extension. Then,

- $\mathrm{Dom}(\alpha)$ *is an open affine subset of* X.
- $\mathcal{O}_X(\mathrm{Dom}(\alpha))$ *is the integral closure of* $K[\alpha]$ *in* $K(X)$.

Proof.

Omitted for lack of time (see Theorem V.10.8 in Lorenzini). □

Corollary

A nonsingular complete curve X/K is the union of two affine open subsets.

Proof.

Take $\alpha \in K(X)$ such that $K(X)/K(\alpha)$ is finite. By the previous theorem and a previous claim, $X = \mathrm{Dom}(\alpha) \cup \mathrm{Dom}(1/\alpha)$. □