Algebraic-Geometric Codes **Fall 2024/5**

Problem Set 4

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Problem 1. *Let $F/\mathbb{F}_q$ be a function field with genus g.*

(a) Prove that its L-polynomial satisfies $L(1) = h$ (where $h = |\mathcal{C}_0|$).

(b) Using the Hasse-Weil Theorem, show that

$$(\sqrt{q}-1)^{2g} \leq h \leq (\sqrt{q}+1)^{2g}.$$

Problem 2. *(a) Compute the Z-function of the curve $y^2 = 2x^4 + x^2 + 1$ over $\mathbb{F}_3$.*

(b) What is the Z-function of the curve over $\mathbb{F}_9$?

Problem 3. *Let $K = \mathbb{F}_q$ be a field with $\mathrm{char} K \neq 2$ and let $f \in K[X]$ be an irreducible polynomial over K of degree $d > 1$. Prove that*

$$\Big| \sum_{x \in \mathbb{F}_q} \mathrm{QR}(f(x)) \Big| \leq O\left((d-1)\sqrt{q}\right)$$

where QR is the quadratic residue character, i.e. $\mathrm{QR}\colon \mathbb{F}_q \to \{-1, 0, 1\}$ is given by

$$\mathrm{QR}(\alpha) = \alpha^{\frac{q-1}{2}} = \begin{cases} 0 & \alpha = 0 \\ 1 & \alpha = \beta^2 \text{ for some } \beta \in \mathbb{F}_q \\ -1 & \text{otherwise} \end{cases}$$

Hint: Consider a corresponding curve.