Local PID and Singularity

Introduction to Algebraic-Geometric Codes. Fall 2019

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Discussion

Let $f \in \bar{K}[x,y]$ irreducible. Let $(a,b) \in Z_f(\bar{K})$ and $M = \langle x-a, y-b\rangle \in \mathrm{Max}(C_f)$ the corresponding ideal via Hilbert's Nullstellensatz. Recall that we identify $(C_f)_M$ with the elements of $\bar{K}(Z_f)$ that are defined at (a,b).

In this unit we establish an algebraic equivalent for the geometric property that (a,b) is a nonsingular point of $Z_f(\bar{K})$. We prove

Theorem

With the notation above,

$$(a,b) \in Z_f(\bar{K}) \text{ is nonsingular} \iff (C_f)_M \text{ is a PID.}$$

We start by proving the following nice basic ring-theoretic claim.

Claim

Let A be a ring. If every prime ideal of A is principal then A is a PID.

The proof makes a nice use of Zorn's lemma but we'll skip it.

We assert that the following claim implies the main theorem.

Claim

Let $f \in \bar{K}[x, y]$ irreducible. Let $(a, b) \in Z_f(\bar{K})$ and $M = \langle x - a, y - b\rangle \in \mathrm{Max}(C_f)$. Then,

$$(a, b) \text{ is nonsingular} \iff M(C_f)_M \text{ is principal.}$$

Proof for Claim $\implies$ Theorem.

f irreducible $\implies \dim(C_f) = 1 \implies \dim((C_f)_M) = 1$. Hence, $\mathrm{Spec}((C_f)_M) = \{\langle 0\rangle, M(C_f)_M\}$. The proof then follows by the previous claim. □

Proof of Claim.

For simplicity, we consider $(a, b) = (0, 0)$. Assume w.l.o.g that $\frac{\partial f}{\partial y}(0, 0) = \delta \neq 0$. Then, in $\bar{K}[x, y]$

$$f(x, y) \in \frac{\partial f}{\partial x}(0, 0)x + \delta y + \langle x, y\rangle.$$

So, there are $g(x, y), h(x, y) \in \bar{K}[x, y]$, $g(0, 0) = 0$ such that

$$f(x, y) = h(x) + y(\delta + g(x, y)).$$

Thus, in C_f

$$0 = h(x) + y(\delta + g(x, y)).$$

□

Proof (cont.)

Thus, in C_f

$$-h(x) = y(\delta + g(x,y)).$$

To conclude this direction, it suffices to show that

$$\delta + g(x,y) \in (C_f)_M^\times.$$

$g(0,0) = 0 \implies g(x,y) \in M = \langle x, y\rangle$. Since $\delta \notin M$ we have $\delta + g(x,y) \notin M$. Thus, $\delta + g(x,y) \in (C_f)_M^\times$. □

Proof (cont.)

In the other direction, Assume $M(C_f)_M = \langle z \rangle$. Since $M = \langle x, y \rangle$ in C_f then in $(C_f)_M$:

$$z = ux + vy$$

$$x = zs$$

$$y = zr$$

for some $u, v, s, r \in (C_f)_M$. Using that $(C_f)_M$ is a domain,

$$us + vr = 1.$$

Thus, one of $s, r \in (C_f)_M^\times$. Say it is s. Note also that $rx = sy$. $\square$

Proof (cont.)

$rx = sy$ in $(C_f)_M$. So,

$$\frac{r_1}{r_2} \cdot \frac{x}{1} = \frac{s_1}{s_2} \cdot \frac{y}{1}$$

for $r_1, r_2, s_1, s_2 \in C_f$ and $r_2, s_2 \notin M$. Since $s \in (C_f)_M^\times$, $s_1 \notin M$ as well. Thus, since M is prime, $r_2 s_1 \notin M$. So, in C_f we have

$$r_1 s_2 x = r_2 s_1 y$$

Let $a(x,y) \in \bar{K}[x,y]$ be such that $a(x,y) + \langle f \rangle = r_1 s_2$.
Let $b(x,y) \in \bar{K}[x,y]$ be such that $b(x,y) + \langle f \rangle = r_2 s_1$.
$r_2 s_1 \notin M \implies b(0,0) \neq 0$. □

Proof (cont.)

In $\bar{K}[x, y]$,

$$f(x, y) \mid a(x, y)x - b(x, y)y$$

So, there exists $g(x, y) \in \bar{K}[x, y]$ such that

$$f(x, y)g(x, y) = a(x, y)x - b(x, y)y$$

What is the coefficient of y? Looking at the LHS

$$\left(f(0,0) + \frac{\partial f}{\partial y}(0,0)y + \cdots \right) \cdot \left(g(0,0) + \frac{\partial g}{\partial y}(0,0)y + \cdots \right)$$

and so, the coefficient is

$$f(0,0)\frac{\partial g}{\partial y}(0,0) + g(0,0)\frac{\partial f}{\partial y}(0,0) = g(0,0)\frac{\partial f}{\partial y}(0,0)$$

□

Proof (cont.)

Looking at the RHS, the coefficient is $b(0,0)$ and so

$$g(0,0)\frac{\partial f}{\partial y}(0,0) = b(0,0) \neq 0$$

In particular,

$$\frac{\partial f}{\partial y}(0,0) \neq 0$$

and so $(0,0)$ is a nonsingular point of $Z_f(\bar{K})$. □

Summary

Let $f \in \bar{K}[x, y]$ *irreducible. Let* $(a, b) \in Z_f(\bar{K})$ *and* $M = \langle x - a, y - b \rangle \in \mathrm{Max}(C_f)$ *the corresponding ideal. Recall that we identify* $(C_f)_M$ *with the elements of* $\bar{K}(Z_f)$ *that are defined at* (a, b)*. Then,*

$$(a, b) \in Z_f(\bar{K}) \text{ is nonsingular} \iff (C_f)_M \text{ is a PID.}$$