Localization of Modules

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Construction ($S^{-1}M$)

Let A be a domain and $S \subset A$ a multiplicative subset of A. Recall the construction of the domain $S^{-1}A$. Let M be an A-module. We now construct an $S^{-1}A$-module which we denote by $S^{-1}M$. To this end, we define a relation on $M \times S$ as follows:

$$(m, s) \equiv (n, t) \iff \exists \sigma \in S \;\; s.t \;\; \sigma(tm - sn) = 0.$$

This is an equivalence relation on $M \times S$. We denote the equivalence class of (m, s) by $\frac{m}{s}$, and let $S^{-1}M$ denote the set of equivalence classes.

Construction ($S^{-1}A$ continued)

We endow the set $S^{-1}M$ with the structure of an $S^{-1}A$-module as follows:

$$\frac{m}{s} + \frac{n}{t} = \frac{tm + sn}{st}$$
$$\frac{a}{s} \cdot \frac{m}{t} = \frac{am}{st}.$$

The $S^{-1}A$-module $S^{-1}M$ is called the module of fractions of M with respect to S.

Definition

Let A be a domain and S a multiplicative subset. Let M, N be a pair of A-modules. An A-module homomorphism $f : M \to N$ induces an $S^{-1}A$-module homomorphism

$$S^{-1}(f) : S^{-1}M \to S^{-1}N$$

that is defined as follows:

$$S^{-1}(f)\left(\frac{m}{s}\right) = \frac{f(m)}{s}.$$

One can verify this is well defined.

Claim

Let A be a domain and S a multiplicative subset. Let $f: M \to N$, $g: N \to Q$ be A-module homomorphism. Then,

$$S^{-1}(g \circ f) = S^{-1}(g) \circ S^{-1}(f)$$

Proof.

By definition,

$$S^{-1}(g \circ f)\left(\frac{m}{s}\right) = \frac{(g \circ f)(m)}{s} = \frac{g(f(m))}{s} = S^{-1}(g)\left(\frac{f(m)}{s}\right).$$

But,

$$\frac{f(m)}{s} = S^{-1}(f)\left(\frac{m}{s}\right).$$

Thus,

$$S^{-1}(g \circ f)\left(\frac{m}{s}\right) = S^{-1}(g) \circ S^{-1}(f)\left(\frac{m}{s}\right)$$

for all $\frac{m}{s} \in S^{-1}M$. □

Claim

Let A be a domain and $S \subseteq A$ multiplicative. Let $f : M \to N$ be an A-module homomorphism. Then,

$$S^{-1}\mathrm{Im}(f) = \mathrm{Im}(S^{-1}(f))$$
$$S^{-1}\ker(f) = \ker(S^{-1}(f))$$

Corollary

Let

$$M' \xrightarrow{f} M \xrightarrow{g} M''$$

be a sequence of A-modules exact at M. Let $S \subset A$ be a multiplicative subset. Then,

$$S^{-1}M' \xrightarrow{S^{-1}(f)} S^{-1}M \xrightarrow{S^{-1}(g)} S^{-1}M''$$

is exact at $S^{-1}M$.

Proof.

$\mathrm{Im}(f) = \ker(g) \implies S^{-1}(\mathrm{Im}(f)) = S^{-1}(\ker(g)) \implies \mathrm{Im}(S^{-1}(f)) = \ker(S^{-1}(g))$. □

Claim

Let A be a domain. Let M be an A-module. Let N be an A-submodule of M. Let $S \subset A$ multiplicative. Then,

$$S^{-1}(M/N) \cong S^{-1}M/S^{-1}N$$

(as $S^{-1}A$-modules)

Proof.

One proof is "by hand": the $S^{-1}A$ isomorphism is going to be

$$\phi : S^{-1}(M/N) \to S^{-1}M/S^{-1}N$$
$$\frac{m+N}{s} \mapsto \frac{m}{s} + S^{-1}N.$$

Verify to yourself that this is a well-defined $S^{-1}A$-module isomorphism. □

Proof.

Another proof is as follows: Consider the exact sequence of A-modules

$$0 \longrightarrow N \longrightarrow M \longrightarrow M/N \longrightarrow 0.$$

By the previous claim, localizing at S gives an exact sequence

$$0 \longrightarrow S^{-1}N \longrightarrow S^{-1}M \longrightarrow S^{-1}(M/N) \longrightarrow 0.$$

By exactness and by the first isomorphism theorem for modules we conclude.

$$S^{-1}(M/N) \cong S^{-1}M/S^{-1}N$$

□

Claim

Let A be a domain. Let M be an A-module. Let N, P be A-submodules of M. Then,

$$S^{-1}(N \cap P) = S^{-1}N \cap S^{-1}P.$$

Proof.

exercise. □

Definition

When $S = A \setminus P$ for $P \in \text{Spec}(A)$, we write M_P for the A_P-module $S^{-1}A$. Similarly, if $f : M \to N$ is an A-module we write f_P for $S^{-1}(f)$.

The following claim states that being the zero module is a local property.

Claim

Let M be an A-module. TFAE

1. $M = (0)$
2. $M_P = (0)$ *for all* $P \in \text{Spec}(A)$
3. $M_P = (0)$ *for all* $P \in \text{Max}(A)$

Proof.

We prove (3) $\implies$ (1). Take $m \in M$ and fix $P \in \mathrm{Max}(A)$. By assumption there exists $s_P \in A \setminus P$ such that

$$s_P \left(\frac{m}{1} - \frac{0}{1} \right) = 0 \quad \implies \quad s_P m = 0.$$

Consider the ideal I of A generated by

$$\{ s_P \mid P \in \mathrm{Max}(A) \}.$$

By construction, I is not contained in any maximal ideal of A and so $1 \in I$. Thus,

$$1 = \sum_P a_P s_P$$

and so

$$m = 1 \cdot m = \sum_P a_P (s_P m) = 0$$

□

The next lemma asserts that being exact is a local property.

Lemma

Let $M' \xrightarrow{f} M \xrightarrow{g} M''$ be a sequence of A-modules. TFAE:

1. $M' \xrightarrow{f} M \xrightarrow{g} M''$ *is exact at* M.
2. $M'_P \xrightarrow{f_P} M_P \xrightarrow{g_P} M''_P$ *is exact at* M_P *for all* $P \in \mathrm{Spec}(A)$.
3. $M'_P \xrightarrow{f_P} M_P \xrightarrow{g_P} M''_P$ *is exact at* M_P *for all* $P \in \mathrm{Max}(A)$.

Proof.

We already proved (1) $\implies$ (2). (2) $\implies$ (3) is trivial. We turn to prove (3) $\implies$ (1). Consider the exact sequence

$$0 \longrightarrow \mathrm{Im}(f) \hookrightarrow \ker(g) \longrightarrow \ker(g)/\mathrm{Im}(f) \to 0.$$

Thus, localizing at $P \in \mathrm{Max}(A)$ yields an exact sequence

$$0 \longrightarrow \mathrm{Im}(f)_P \hookrightarrow \ker(g)_P \longrightarrow (\ker(g)/\mathrm{Im}(f))_P \to 0.$$

Now,

$$(\ker(g)/\mathrm{Im}(f))_P \cong \ker(g)_P/\mathrm{Im}(f)_P = \ker(g_P)/\mathrm{Im}(f_P).$$

By assumption $\ker(g_P)/\mathrm{Im}(f_P) = (0)$. By the previous claim, $\ker(g)/\mathrm{Im}(f) = (0)$. □

Corollary

Let $f : M \to N$ be a map of A-modules. The map f is injective (surjective) if and only if the maps f_P are injective (surjective) for all $P \in \mathrm{Max}(A)$.

Proof.

Simply "encode" injectivity (surjectively) using exact sequences. Indeed, f is injective if and only if the sequence

$$0 \longrightarrow M \xrightarrow{f} N$$

is exact at M. □

Lemma

Let A be a domain. Then,

$$A = \bigcap_{P \in \mathrm{Spec}(A)} A_P = \bigcap_{P \in \mathrm{Max}(A)} A_P.$$

Proof.

Clearly, the inclusions $\subseteq$ hold. Denote the A-module $\bigcap_{P \in \mathrm{Max}(A)} A_P$ by M. By the previous claim, it suffices to show that for every $Q \in \mathrm{Max}(A)$, the inclusion map $i_Q : A \hookrightarrow M$ is surjective.
Now, for any multiplicative subset S of A,

$$S^{-1}M = S^{-1}\left(\bigcap_{P \in \mathrm{Max}(A)} A_P\right) \subseteq \bigcap_{P \in \mathrm{Max}(A)} S^{-1}(A_P).$$

□

Proof.

Also, for $Q \in \mathrm{Max}(A)$, $(A_Q)_Q = A_Q$. Thus,

$$A_Q \hookrightarrow M_Q \subseteq \bigcap_{P \in \mathrm{Max}(A)} (A_P)_Q \subseteq A_Q.$$

□

As a corollary we obtain that being integrally closed is a local property.

Corollary

Let A be a domain. TFAE:

1. *A is integrally closed;*
2. *A_P is integrally closed for all $P \in \operatorname{Spec}(A)$;*
3. *A_P is integrally closed for all $P \in \operatorname{Max}(A)$.*

Proof.

We have already proved (1) $\implies$ (2) and (2) $\implies$ (3) is trivial. As for (3) $\implies$ (1), take $b/c \in K$ that is integral over A. For any $P \in \operatorname{Max}(A)$, b/c is integral over A_P and so, by assumption, $b/c \in A_P$.Hence, $b/c \in \cap_{P \in \operatorname{Max}(A)} A_P = A$. □