Free Probability and Ramanujan Graphs - HW #4

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Exercises with * will be graded. Submission in singles or pairs. Contact Gal for questions and clarifications.

1. * Let $\sigma \in NC(n)$ be an interval partition; that is, all blocks consist of consecutive numbers (example: $(123, 45, 67)$ is an interval partition in $NC(7)$). Prove that for any $\pi \in NC(n)$, the join $\sigma \vee \pi$ is the same in $NC(n)$ and in $P(n)$.

2. Let $P(n)$ be the lattice of all partitions of $[n]$. Prove that $\mu_{P(n)}(0_n, 1_n) = (-1)^{n-1}(n-1)!$ (hint: follow the calculation of $\mu_{P(n)}$ from class, and use the factorization of $P(n)$ to intervals).

3. Prove the following proposition from the lecture: Let $(f_n)_{n\geq 1}$ be a multiplicative family on NC, and $(F_n)_{n\geq 1}$ be a multiplicative family on $NC^{(2)}$. Prove that $(f_n * F_n)_{n\geq 1}$ is a multiplicative family on NC.

Interlacing Polynomials

In this part we prepare for the final stage of the course, in which we translate ideas from free probability to finite matrices, and their characteristic polynomials.

We denote by $\mathbb{P}(n)$ the set of all degree n real-rooted polynomials with a positive leading coefficient. We write $\mathsf{maxroot}(p)$ for the maximal root of a polynomial $p(x) \in \mathbb{P}(n)$. For two polynomials $p(x) = \prod_{i=1}^{n}(x - \alpha_i)$ and $q(x) = \prod_{i=1}^{n}(x - \beta_i)$ (where the roots are sorted such that $\alpha_i \geq \alpha_{i+1}$, and same for β_i) we say that "$p(x)$ interlaces $q(x)$", and denote $p(x) \to q(x)$, if

$$\alpha_n \leq \beta_n \leq \alpha_{n-1} \leq \beta_{n-1} \leq \cdots \leq \alpha_1 \leq \beta_1.$$

4. * Let $r(x), p_1(x), p_2(x) \in \mathbb{P}(n)$ be such that $r(x) \to p_1(x)$ and $r(x) \to p_2(x)$.

   (a) Prove that $r(x) \to p_1(x) + p_2(x)$.

   (b) Prove that there exists $i \in \{1, 2\}$ such that $\mathsf{maxroot}(p_i) \leq \mathsf{maxroot}(p_1 + p_2)$.