The Krull Dimension of a Ring

Introduction to Algebraic-Geometric Codes. Fall 2019

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Definition

Let A be a ring. The set of all prime ideals in A is denoted by $\mathrm{Spec}(A)$.

Remark

- $\langle 1 \rangle = A \notin \mathrm{Spec}(A)$.
- $\langle 0 \rangle \in \mathrm{Spec}(A) \iff A$ *is a domain.*

Definition

Let A be a ring. A sequence $P_n \subset \cdots \subset P_1 \subset P_0$ of distinct prime ideals in A is called a chain of prime ideals of length n.

Definition

Let A be a ring and $P \in \mathrm{Spec}(A)$. The height of P, denoted by $\mathrm{ht}(P)$ is the supremum of lengths of chains of prime ideals in A ending with $P_0 = P$.

Definition

Let A be a ring. The Krull dimension of A is defined by

$$\dim(A) = \sup\{\mathrm{ht}(P) \mid P \in \mathrm{Spec}(A)\}.$$

Example

- Any field has dimension 0. Indeed, $\mathrm{Spec}(\text{field}) = \{\langle 0 \rangle\}$.
- A domain of dimension 0 is a field.
- Let k be a field. $k[x]/\langle x^2 \rangle$ is a non-domain with dimension 0.
- $\dim(\mathbb{Z}) = 1$. For ≥ 1 consider $\langle 0 \rangle \subset \langle 2 \rangle$.
- Let k be a field. Then, $\dim(k[x]) = 1$. For ≥ 1 consider $\langle 0 \rangle \subset \langle x \rangle$.
- More generally, one can prove that $\dim(k[x_1, \ldots, x_n]) = n$.
- We will now prove that a PID which is not a field has dimension 1.
- $\dim(\mathbb{Z}[x]) = 2$. The $\geq$ follows by $\langle 0 \rangle \subset \langle 2 \rangle \subset \langle 2, x \rangle$.

Remark

Let A be a domain. Then, $\dim(A) = 1 \iff$ *every nonzero prime ideal is maximal* $+$ $\mathrm{Spec}(A) \neq \{\langle 0 \rangle\}$.

Claim

Every PID that is not a field has dimension 1*.*

The proof readily follows by the following claim that is left as an exercise.

Claim

Let A *be a domain,* $P = \langle p \rangle, Q = \langle q \rangle$ *be two distinct, nontrivial, prime ideals. Then,* $P \not\subset Q$*.*

Two neat propositions that we will not need but good to know.

Proposition

Let R be a domain. Then,

$$R \text{ is a UFD} \iff \text{every height 1 prime ideal is principal}$$

Proposition

Noetherian + Dimension 1 + UFD $\iff$ PID.

Proposition (Main result of this unit)

Let B/A be an integral extension. Assume B is a domain. Then,

$$\dim(A) = 1 \implies \dim(B) = 1.$$

This type of result holds for higher dimensions. For instance (without a proof):

Theorem

Let B/A be an integral extension. Assume A, B are noetherian domains. Then, $\dim(B) = \dim(A)$.

Claim

Let B/A be a ring extension. Assume B is a domain. Let $P_B \in \mathrm{Spec}(B)$. Define $P_A = P_B \cap A$. Then, $P_A \in \mathrm{Spec}(A)$.

Proof.

Observe that $A/P_A \hookrightarrow B/P_B$ via $a + P_A \mapsto a + P_B$. Thus, A/P_A is a domain and so $P_A \in \mathrm{Spec}(A)$. □

Claim

Let B/A be an integral extension. Assume B is a domain. Let $P_B \in \text{Spec}(B) \setminus \{\langle 0 \rangle\}$. Then, $P_A = P_B \cap A \in \text{Spec}(A) \setminus \{\langle 0 \rangle\}$.

Proof.

Take $0 \neq \alpha \in P_B$. As α integral over A,

$$\alpha^n + a_{n-1}\alpha^{n-1} + \cdots + a_1\alpha + a_0 = 0,$$

where $a_i \in A$. Since B is a domain, we may assume $a_0 \neq 0$. We see that $a_0 \in P_A$. □

Corollary

Let B/A be an integral extension. Assume B is a domain and that $\dim(A) = 1$. Let $\langle 0 \rangle \neq P_B \in \mathrm{Spec}(B)$ and define $P_A = P_B \cap A$. Then, $P_A \in \mathrm{Max}(A)$.

Proof.

We proved that $P_A \in \mathrm{Spec}(A) \setminus \{\langle 0 \rangle\}$. So, $\mathrm{ht}(P_A) \geq 1$. But $\dim(A) = 1$ and so $\mathrm{ht}(P_A) = 1$. Hence, $P_A \in \mathrm{Max}(A)$. □

Claim

Let B/A be an integral extension. Assume B is a domain and that $\dim(A) = 1$. Let $\langle 0 \rangle \neq P_B \in \mathrm{Spec}(B)$ and define $P_A = P_B \cap A$. Then, $P_B \in \mathrm{Max}(B)$.

Proof.

Observe that B/P_B is a domain that contains an isomorphic copy of the field A/P_A via $a + P_A \mapsto a + P_B$. Take $\beta + P_B \in B/P_B$ with $\beta \in B \setminus P_B$. Then, $\exists a_i \in A$ s.t. in B

$$\beta^n + a_{n-1}\beta^{n-1} + \cdots + a_0 = 0$$

Thus, in B/P_B,

$$(\beta + P_B)^n + (a_{n-1} + P_B)(\beta + P_B)^{n-1} + \cdots + (a_0 + P_B) = 0.$$

□

Proof.

$$(\beta + P_B)^n + (a_{n-1} + P_B)(\beta + P_B)^{n-1} + \cdots + (a_0 + P_B) = 0.$$

Using that B/P_B is a domain, we may assume $a_0 + P_B \neq 0$. Thus,

$$(\beta + P_B) \cdot (\gamma + P_B) = a_0 + P_B.$$

Since A/P_A is a field, $a_0 + P_B$ is a unit of B/P_B and so $\beta + P_B$ is invertible in B/P_B. □

Recall the proposition we were set to prove.

Proposition

Let B/A be an integral extension. Assume B is a domain. Then,

$$\dim(A) = 1 \implies \dim(B) = 1.$$

Proof.

We need to show that every $\langle 0 \rangle \neq P_B \in \mathrm{Spec}(B)$ is maximal, which is exactly what we proved. $\square$

We are not yet ready to prove the following claims - we will need the notion of localization from commutative algebra, but these are good to have in mind. We'll prove them in the recitations / assignments.

Claim

Let A be a domain of dimension 1. Then $A[y]$ has dimension 2 or 3. Moreover, if A is a PID then $\dim(A[y]) = 2$.

Claim (very important for us)

Let K be a field. Let $f \in K[x,y]$ irreducible. Then, $K[x,y]/\langle f \rangle$ has dimension 1.