Graph Partitioning and Cheeger's Inequality

Following Spielman, Chapters 20,21

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Overview

Isoperimetry and λ_2

Definition

Let $G = (V, E)$ be an undirected graph. For $S \subseteq V$ we define the boundary of S by

$$\partial(S) = \{uv \in E \mid u \in S, v \notin S\}.$$

The isoperimetric ratio of $S \neq \emptyset$ is defined by

$$\theta(S) = \frac{|\partial(S)|}{|S|}.$$

The isoperimetric ratio of G is given by

$$\theta_G = \min_{\emptyset \neq S \subset V} \max(\theta(S), \theta(V \setminus S)) = \min_{0 < |S| \leq \frac{n}{2}} \theta(S).$$

Isoperimetry and λ_2

Theorem

For every $S \subset V$,

$$\theta(S) \geq \lambda_2(1 - s),$$

where $s = \frac{|S|}{|V|}$. *In particular,* $\theta_G \geq \frac{\lambda_2}{2}$.

Extra space for the proof

Conductance

Definition

Let $G = (V, E)$ be an undirected graph. For $\emptyset \neq S \subset V$ we define the conductance of S by

$$\phi(S) = \frac{|\partial(S)|}{\min(d(S), d(V \setminus S))},$$

where $d(S) = \sum_{v \in S} \deg(v)$. The conductance of G is given by

$$\phi_G = \min_{\emptyset \neq S \subset V} \phi(S).$$

The normalized Laplacian

Definition

Let $G = (V, E)$ be an undirected graph. We define the normalized Laplacian of G by

$$\mathbf{N} = \mathbf{D}^{-1/2}\mathbf{L}\mathbf{D}^{-1/2}.$$

We reserve $0 = \nu_1 \leq \nu_2 \leq \cdots \leq \nu_n$ for its eigenvalues.

Recall that $\mathbf{A} = \mathbf{D}^{-1/2}\mathbf{M}\mathbf{D}^{-1/2}$ and so $\mathbf{N} = \mathcal{I} - \mathbf{A}$. Thus, $\nu_i = 1 - \omega_i$. The corresponding eigenvector to $\nu_1 = 0$ is $\sqrt{\mathbf{d}}$ as indeed

$$\mathbf{N}\sqrt{\mathbf{d}} = \mathbf{D}^{-1/2}\mathbf{L}\mathbf{1} = \mathbf{0}.$$

The normalize Laplacian

Definition

The generalized Rayleigh quotient of $\mathbf{y}$ (with respect to $\mathbf{L}$) is defined by

$$\frac{\mathbf{y}^T\mathbf{L}\mathbf{y}}{\mathbf{y}^T\mathbf{D}\mathbf{y}}$$

Note that by setting $\mathbf{x} = \mathbf{D}^{1/2}\mathbf{y}$ we get

$$\frac{\mathbf{y}^T\mathbf{L}\mathbf{y}}{\mathbf{y}^T\mathbf{D}\mathbf{y}} = \frac{\mathbf{x}^T\mathbf{N}\mathbf{x}}{\mathbf{x}^T\mathbf{x}}.$$

Thus,

$$\nu_2 = \min_{\mathbf{y}\perp\mathbf{d}} \frac{\mathbf{y}^T\mathbf{L}\mathbf{y}}{\mathbf{y}^T\mathbf{D}\mathbf{y}}.$$

PD norms

Give an $n \times n$ PD matrix $\mathbf{A}$, we define the $\mathbf{A}$-norm on $\mathbb{R}^n$ by

$$\|\mathbf{x}\|_{\mathbf{A}} = \sqrt{\mathbf{x}^T \mathbf{A} \mathbf{x}} = \|\mathbf{A}^{1/2}\mathbf{x}\|_2.$$

The Rayleigh quotient of $\mathbf{x}$ measures the ratio of two norms of $\mathbf{x}$,

$$\frac{\mathbf{x}^T \mathbf{L} \mathbf{x}}{\mathbf{x}^T \mathbf{x}} = \frac{\|\mathbf{x}\|_{\mathbf{L}}}{\|\mathbf{x}\|_{\mathcal{I}}}$$

whereas the generalized Rayleigh quotient is given by

$$\frac{\|\mathbf{x}\|_{\mathbf{L}}}{\|\mathbf{x}\|_{\mathbf{D}}}.$$

The normalize Laplacian

Theorem

For every $\emptyset \neq S \subset V$,

$$\frac{|\partial(S)|d(V)}{d(S)d(V \setminus S)} \geq \nu_2.$$

In particular,

$$\phi_G \geq \frac{\nu_2}{2}$$

You will probably be asked to prove this in the problem set.

Cheeger's Inequality

Theorem (Cheeger's inequality)

$$\frac{\nu_2}{2} \leq \phi_G \leq \sqrt{2\nu_2}$$

A Lower Bound for the
Smallest Eigenvalue of the Laplacian

JEFF CHEEGER

Various authors have studied the geometrical and topological significance of the spectrum of the Laplacian Δ^2, on a Riemannian manifold. (The excellent survey article of Berger [2] contains background, references, and open problems.) The purpose of this note is to give a lower bound for the smallest eigenvalue $\lambda > 0$ of Δ^2 applied to functions. The bound is in terms of a certain global geometric invariant, essentially the constant in the isoperimetric inequality. The technique works for compact manifolds of arbitrary dimension with or without boundary.

The author wishes to thank J. Simons for helpful conversations and in particular for suggesting the importance of understanding the following example of E. Calabi. Consider the "dumbbell" manifold homeomorphic to S^2, shown in Fig. 1. The pipe connecting the two halves is to be thought of as having fixed length l and variable radius r.

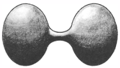


Fig. 1

One sees that $\lambda \to 0$ as $r \to 0$. Calabi's original argument involved consideration of the heat equation, $\frac{-\partial T}{\partial t} = \Delta^2 T$.

A somewhat more direct argument is as follows: Let f be a function which is equal to c on the right-hand bulb, $-c$ on the left-hand bulb and

Cheeger's Inequality

We will follow a proof due to Trevisan. Let $\mathbf{y} \perp \mathbf{d}$ with

$$\nu_2' = \frac{\mathbf{y}^T \mathbf{N} \mathbf{y}}{\mathbf{y}^T \mathbf{y}}.$$

Relabel the vertices so that $\mathbf{y}(1) \leq \mathbf{y}(2) \leq \cdots \leq \mathbf{y}(n)$. It will be convenient to "center $\mathbf{y}$ around $\mathbf{d}$", namely, for j being the least number for which

$$\sum_{i=1}^{j} \mathbf{d}(i) \geq \frac{d(V)}{2},$$

define $\mathbf{z} = \mathbf{y} - \mathbf{y}(j)\mathbf{1}$.

Cheeger's Inequality

Claim

We have that

$$\frac{\mathbf{z}^T \mathbf{L} \mathbf{z}}{\mathbf{z}^T \mathbf{D} \mathbf{z}} \leq \frac{\mathbf{y}^T \mathbf{L} \mathbf{y}}{\mathbf{y}^T \mathbf{D} \mathbf{y}}.$$

We further normalize $\mathbf{z}$ so that $\mathbf{z}(1)^2 + \mathbf{z}(n)^2 = 1$.

Cheeger's Inequality

Proposition

There exists τ *for which* $S_\tau = \{i \in [n] \mid z(i) < \tau\}$ *satisfies*

$$\phi(S_\tau) \le \sqrt{2 \cdot \frac{\mathbf{z}^T \mathbf{L} \mathbf{z}}{\mathbf{z}^T \mathbf{D} \mathbf{z}}}.$$

The key idea in Trevisan's proof is to sample τ from the distribution supported on $[z(1), z(n)]$ with probability density $2|t|$. Namely, for every $z(1) \le a \le b \le z(n)$,

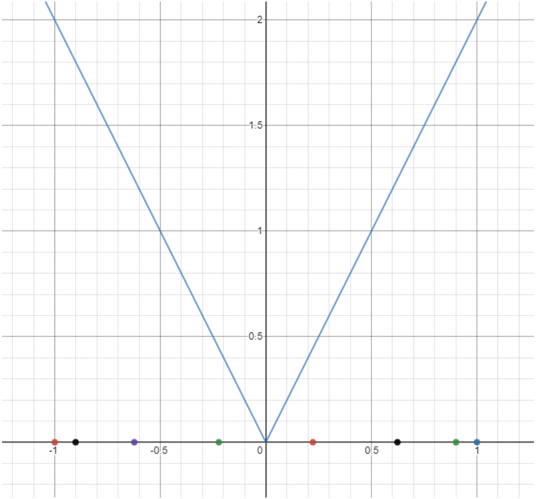
$$\Pr[\tau \in [a, b]] = \int_a^b 2|t| dt.$$

Cheeger's Inequality

Claim

This is indeed a probability distribution.

Cheeger's Inequality



Cheeger's Inequality

Claim

$$\mathbb{E}[\min(d(S_\tau), d(V \setminus S_\tau))] = \mathbf{z}^T \mathbf{D} \mathbf{z}.$$

Extra space for the proof

Cheeger's Inequality

Claim

$$\mathbb{E}[\partial(S_\tau)] \leq \sum_{uv \in E} |\mathbf{z}(u) - \mathbf{z}(v)|(|\mathbf{z}(u)| + |\mathbf{z}(v)|).$$

Extra space for the proof

Cheeger's Inequality

Claim

$$\begin{aligned}\mathbb{E}[\partial(S_\tau)] &\leq \sqrt{2}\cdot\sqrt{\mathbf{z}^T\mathbf{L}\mathbf{z}}\cdot\sqrt{\mathbf{z}^T\mathbf{D}\mathbf{z}} \\ &\leq \sqrt{2\nu_2'}\cdot\mathbf{z}^T\mathbf{D}\mathbf{z}\end{aligned}$$

Cheeger's Inequality

To recap,

$$\mathbb{E}[\partial(S_\tau)] \leq \sqrt{2\nu_2'} \cdot \mathbf{z}^T \mathbf{D} \mathbf{z},$$
$$\mathbb{E}[\min(d(S_\tau), d(V \setminus S_\tau))] = \mathbf{z}^T \mathbf{D} \mathbf{z}.$$

Hence,

Extra space for the proof