Spectral Graph Theory **Winter 2020**

Final Exam

Publish Date: January 31, 2021 *Due Date: February 4, 2021 (all day long)*

Note: Submissions are *not* in pairs. You may not consult with anyone nor use any material other than what has been taught in class. You may also use any statement made in the assignments (even if you haven't solved it). Your solutions are to be typed and submitted via Moodle as a PDF. If you have any question please send an email to both Shir and Gil.

Problem 1 Let $G = (V, E)$ be an undirected graph. Define the graph $H = (E, E')$ that has an edge $e' = \{e_1, e_2\} \in E'$ for every two distinct edges $e_1, e_2 \in E$ of G with a common vertex. Prove that all eigenvalues of the adjacency matrix of H are bounded below by -2.

Problem 2 Let $G = (V, E)$ be a d-regular graph on n vertices that is 3-colorable and such that there is a 3-coloring in which the color classes have equal size $n/3$. Let A be the normalized adjacency matrix of G.

1. Prove that A has at least two eigenvalues which are smaller than or equal to $-1/2$, that is, $\mu_{n-1} \leq -1/2$.
2. Give an example in which the bound the tight. That is, for every n that is divisible by 3, give an example of a regular graph satisfying the conditions of the claim such that $\mu_{n-1} = -1/2$.
3. Show that the converse is false. That is, for every large enough n, give an example of a regular graph that is not 3-colorable but such that at least two eigenvalues of the normalized adjacency matrix are $\leq -1/2$.

Problem 3 Let $G = (V, E)$ be a $(1 - \omega)$ spectral expander, and let $f : V \to \{0, 1\}$ be such that $f(v) = 1$ for exactly half the vertices $v \in V$. Let $1 \leq i < j < k \leq t$. Give the best bound you can (in terms of ω, i, j, k) on

$$\left| \mathbb{E}_{(v_1, v_2, \ldots, v_t)} \left[(-1)^{f(v_i) + f(v_j) + f(v_k)} \right] \right|,$$

where $(v_1, v_2, \ldots, v_t)$ are the vertices on a random walk on G.

Problem 4 Let $G = (V, E)$ be an undirected, unweighted, connected graph. Let T be a spanning tree of G. For vertices $u, v \in V$ denote by $\Delta_T(u, v)$ the path length between u and v in T (measured in edges). Prove that

$$\mathbb{E}_{i \in V} \left[\alpha_i(\mathbf{L}_T^+ \mathbf{L}_G) \right] = \frac{|E|}{|V|} \cdot \mathbb{E}_{uv \in E} [\Delta_T(u, v)],$$

where $\alpha_1(\mathbf{A}), \ldots, \alpha_n(\mathbf{A})$ are the eigenvalues of an $n \times n$ matrix $\mathbf{A}$. Note that in the course we did not discuss eigenvalues of non symmetric matrices, and thus any property (including existence) of the LHS should be proved carefully.

Problem 5 Let $G=(V,E)$ be a d-regular connected graph on n vertices. In this question you will be asked to construct a spectral approximation $\widetilde{G^2}$ of the square G^2. To this end, let H be a c-regular $(1-\omega)$-spectral expander on d vertices for some $\omega < \omega_0$, where $\omega_0 < 1$ is constant. Observe that in G^2, for every $v \in V$ the neighbors of v in G, denoted by $\Gamma_G(v)$, are connected by the clique with self loops. This inspires the following construction of $\widetilde{G^2}$. Instead of having a clique on every neighborhood $\Gamma_G(v)$, "put" the expander H.

Formally, let $\pi_G : V \times [d] \to V \times [d]$ be the edge rotation map of G, and $\pi_H : [d] \times [c] \to [d] \times [c]$ the edge rotation map of H. The graph $\widetilde{G^2}$ is defined on the vertex set V, and has the following edges. For every $v \in V$, $i \in [d]$ and $j \in [c]$ we add the edge $\dot{\pi}_G(v,i), \dot{\pi}_G(v, \dot{\pi}_H(i,j))$ (the same edge may be added more than once). Define

$$\mathbf{L}_{H,v} = \sum_{i=1}^{d} \sum_{j=1}^{c} \mathbf{L}_{\dot{\pi}_G(v,i), \dot{\pi}_G(v, \dot{\pi}_H(i,j))},$$

and note that

$$\mathbf{L}_{\widetilde{G^2}} = \sum_v \mathbf{L}_{H,v}.$$

The notion of approximation that we will use in this question is somewhat different than the one used in class. Let $\mathbf{A}, \mathbf{B}$ be two symmetric matrices. We write $\mathbf{A} \approx_\varepsilon \mathbf{B}$ if $e^{-\varepsilon}\mathbf{B} \preccurlyeq \mathbf{A} \preccurlyeq e^{\varepsilon}\mathbf{B}$.

1. Prove that $\mathbf{A} \approx_\varepsilon \mathbf{B} \iff \mathbf{B} \approx_\varepsilon \mathbf{A}$.
2. Assume that $\mathbf{A}, \mathbf{A}', \mathbf{B}, \mathbf{B}'$ are PSD satisfying $\mathbf{A} \approx_{\varepsilon_1} \mathbf{A}'$, $\mathbf{B} \approx_{\varepsilon_2} \mathbf{B}'$. For what $\varepsilon = \varepsilon(\varepsilon_1, \varepsilon_2)$ can you prove that $\mathbf{A} + \mathbf{B} \approx_\varepsilon \mathbf{A}' + \mathbf{B}'$?
3. Prove that for every $v \in V$, $\mathbf{N}_{H,v} \approx_{O(\omega)} \mathbf{N}_{J,v}$, where $\mathbf{N}_{J,v}$ is the normalized Laplacian of the clique with self loops on $\Gamma_G(v)$, and all other vertices being isolated. Recall that the definition of the normalized Laplacian given in class needs a slight extension to graphs with isolated vertices. We leave it to you to figure out how.
4. Prove that $\mathbf{N}_{\widetilde{G^2}} \approx_{O(\omega)} \mathbf{N}_{G^2}$.