Spectral Graph Theory **Winter 2020**

Problem Set 4

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Problem 4.1 Let $S \in GL_2(\mathbb{Z})$, S has all non negative entries, $\det(S) = 1$, $S \neq S^T$. Define the graph G on the vertices $V = \mathbb{Z}^2 \setminus \{(0,0)\}$, where the vertex (x, y) is connected to

$$S(x,y), S^T(x,y), S^{-1}(x,y), (S^T)^{-1}(x,y).$$

Prove that for every finite $A \subset V$ it holds that $\frac{|E(A,\overline{A})|}{|A|} \geq 1$.

Problem 4.2 Prove the following generalization of Cheeger's inequality. For every d-regular graph $G = (V, E)$ with a countably infinite set of vertices we have

$$\inf_{A:|A|<\infty} \frac{|E(A,\overline{A})|}{|A|} \leq \sqrt{2 \cdot d \cdot \lambda_1(G)}.$$

Hint: Given a function $f : V \to \mathbb{R}$, consider the family of subsets $S_t := \{v \in V \mid f^2(V) > t\}$ and use integration (w.r.t t) to deduce the desired inequality.

Problem 3.3 In this question you will prove that Cayley graphs coming from Abelian groups cannot yield good expanders. Quantitatively, to achieve constant spectral gap the degree must be at least logarithmic in the number of vertices. Formally, let Γ be an abelian group of size n and $S \subseteq \Gamma$ closed to inverses (namely, $s \in S \implies -s \in S$). Let $G = G(\Gamma, S)$ be the corresponding Cayley graph. Prove that if $\gamma(G) \geq c$ for some constant $c > 0$ then $|S| = \Omega(\log n)$.

Problem 3.4 Let $G = (V, E)$ be a $(1 - \omega)$ spectral expander. Let $f : V \to \{0, 1\}$ and denote by $\mu = |\mathbf{E}_{v\sim V}[(-1)^{f(v)}]|$ (that is, μ is the bias of the labelling given by f). Define a distribution U_t by sampling t vertices $v_1, \ldots, v_t$ uniformly and independently at random. It is straightforward to prove that the bias of the parity of the corresponding bits decreases exponentially with t, that is,

$$\left|\mathbf{E}_{(v_1,\ldots,v_t)\sim U_t}\left[(-1)^{f(v_1)+\cdots+f(v_t)}\right]\right| = \mu^t.$$

In this question, you will prove that an expander random walk has almost the same effect. Formally, let RW_t be the distribution defined by random walks, $(v_1, v_2, \ldots, v_t)$, in G, (that is, v_1 is sampled uniformly at random from V, and for $i > 1$, v_i is sampled uniformly at random from the neighbors of v_{i-1}).

Prove that

$$\left|\mathbf{E}_{(v_1,\ldots,v_t)\sim RW_t}\left[(-1)^{f(v_1)+\cdots+f(v_t)}\right]\right| \leq (O(\mu^2 + \omega))^{t/4}.$$

Problem 3.5 In the lectures, we devised an error reduction technique using expander random walks. Roughly we presented a technique that given an algorithm A with error probability $\leq c$, running time t and randomness complexity r, outputs an algorithm $\tilde{A}$, solving the same problem, with error probability $\leq \varepsilon$, running time $O(\log \frac{1}{\varepsilon} \cdot t)$ and randomness $O(r + \log \frac{1}{\varepsilon})$.

Devise another error reduction technique, which will have *no cost* in randomness complexity though a poly$(\frac{1}{\varepsilon})$ multiplicative cost in time complexity. Moreover, your error reduction technique should work for two-sided error randomized algorithms.

Guidance: instead of taking a random walk, sample a vertex at random and consider its neighbors.