Expander Graphs and Ramanujan Graphs

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Outline

Expanders graphs

Expander graphs are low-degree regular graphs that are nonetheless well-connected:

- A set of vertices have many neighbors outside the set.
- There are no small cuts.
- Random walks mix quickly.
- The number of edges between two sets is "as expected" in a random graph (regular, with the same degree).

Many applications!

- Error correcting codes
- Derandomization
- PCP
- Cryptography,
- Data structures, etc.

Spectral expanders

There are several "flavors" of expanders: edge expanders, vertex expanders, and spectral expanders, which are equivalent up to constants. We focus on spectral expanders.

Definition

Let $G = (V, E)$ be an undirected graph. The spectral gap of G is defined by

$$\gamma(G) = 1 - \omega(G),$$

where, recall, $\omega(G) = \max(|\omega_2(G)|, |\omega_n(G)|)$.

We say that G is a γ-spectral expander if $\gamma(G) \geq \gamma$.

Recall that the spectral gap is related to the rate of convergence of a random walk.

Spectral norm

Definition

Let $\mathbf{A}$ be a real matrix. The spectral norm of $\mathbf{A}$, denoted by $\|\mathbf{A}\|$, is given by

$$\|\mathbf{A}\| = \max_{0 \neq \mathbf{x} \in \mathbb{R}^n} \frac{\|\mathbf{A}\mathbf{x}\|}{\|\mathbf{x}\|} = \max_{0 \neq \mathbf{x} \in \mathbb{R}^n} \frac{\sqrt{\mathbf{x}^T \mathbf{A}^T \mathbf{A}\mathbf{x}}}{\sqrt{\mathbf{x}^T \mathbf{x}}} = \sqrt{\max_{0 \neq \mathbf{x} \in \mathbb{R}^n} \frac{\mathbf{x}^T \mathbf{A}^T \mathbf{A}\mathbf{x}}{\mathbf{x}^T \mathbf{x}}}.$$

Geometrically, $\|\mathbf{A}\|$ measures the largest "stretch" $\mathbf{A}$ can have (ignoring a possible flip of the direction).

Spectral norm

Lemma

We have the following properties of the spectral norm.

- Sub-additivity: $\|\mathbf{A}+\mathbf{B}\| \leq \|\mathbf{A}\| + \|\mathbf{B}\|$.
- Sub-multiplactivity: $\|\mathbf{AB}\| \leq \|\mathbf{A}\|\|\mathbf{B}\|$.
- $\|a\mathbf{A}\| \leq |a| \ \|\mathbf{A}\|$.

Lemma

The spectral norm of **A** equals to the square root of the largest eigenvalue of $\mathbf{A}^T\mathbf{A}$. In particular, when **A** is symmetric,

$$\|\mathbf{A}\| = \max\left\{|\alpha| : \alpha \in \mathrm{Spec}(\mathbf{A})\right\}.$$

Another view on spectral expanders

Lemma

Let $G = (V, E)$ be an undirected regular graph. Then, G is a γ-spectral expander if and only if

$$\mathbf{W}_G = \mathbf{J} + \mathbf{E},$$

where $\mathbf{J}$ stands for the $n \times n$ all $\frac{1}{n}$ matrix, and $\|\mathbf{E}\| \leq 1 - \gamma = \omega$.

Outline

The expander mixing lemma

Given two sets S, T of vertices, we denote

$$e(S, T) = \{uv \in E \mid u \in S, v \in T\}.$$

Lemma (The expander mixing lemma)

Let G be a d-regular $\gamma = 1 - \omega$ spectral expander on n vertices. Let $S, T \subseteq V$ be sets of density α, β respectively. Then,

$$\left| \frac{|e(S, T)|}{nd} - \alpha\beta \right| \leq \omega\sqrt{\alpha\beta}.$$

Hitting property of expander walks

Theorem

Let $G = (V, E)$ be a d-regular $\gamma = 1 - \omega$ spectral expander. Let $v_1, \ldots, v_t$ be a random walk in which v_1 is sampled uniformly at random from V. Then, for every $B \subseteq V$ having density μ,

$$\Pr\left[\{v_1, \ldots, v_t\} \subseteq B\right] \leq (\mu + \omega)^t.$$

Outline

Ramanujan graphs

A natural question is how large can we make γ (equivalently, small $\omega = \frac{\mu}{d}$) as a function of d? The Alon-Boppana bound states that

$$\mu(G) \geq 2\sqrt{d-1}\left(1 - \frac{2}{k+1}\right)$$

where $k = \operatorname{diam}(G)$. Remarkably, this is tight, namely, there are graphs with

$$\mu \leq 2\sqrt{d-1}.$$

Graphs meeting this bound are called Ramanujan graphs.

Ramanujan graphs

Where does this $2\sqrt{d-1}$ expression come from?

The ultimate d-regular expander is the d-regular infinite tree, whose μ equals $2\sqrt{d-1}$. So, in a sense, an expander is a finite approximation of the d-regular infinite tree.

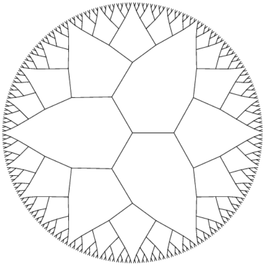

Figure: Infinite 3-regular tree.

Ramanujan graphs

Margulis (1988) and Lubotzky, Phillips and Sarnak (1988) gave the first construction of an infinite family of Ramanujan graphs. The construction has two drawbacks:

1. It has degree $p+1$ for a prime p (later results generalizes that to p^k+1).
2. It is based on extremely heavy mathematics (hence, challenging to understand and "play" with).

Are Ramanujan graphs "special"? Unclear. Friedman (2008) proved that for a "random" d-regular graph, w.h.p,

$$\mu \leq 2\sqrt{d-1} + o_n(1).$$

Ramanujan graphs

Bilu and Linial (2006) gave a construction achieving

$$\mu = O\left(\sqrt{d \cdot \log^3 d}\right).$$

though it has the downside of being “weakly” explicit.

Strongly explicit constructions achieve $\mu = d^{\frac{1}{2}+o(1)}$. Recall that the loose use of the term “expanders” refers to $\mu \leq \alpha d$ for some constant $\alpha < 1$.

A conjecture of Bilu-Linial, if holds, yields the existence of Ramanujan graphs of all degrees. Their conjecture was proved by Marcus, Spielman and Srivastava (2014) for bipartite graphs, yielding the existence of bipartite Ramanujan graphs of all degrees. This is the main topic of our seminar.