Introduction to Algebraic-Geometric Codes **Spring 2019**

Exercise 3

Publish Date: 20 March 19 *Due Date: 11 April 19*

Exercise 3.1. *Let R be a UFD, and K be its fraction field. Let $\alpha \in K$ and $f \in K[x]$, prove that:*

(a) $\alpha \in R^* \iff ord_p(\alpha) = 0$ *for all prime elements* $p \in R$.

(b) $cont(f) \in R \iff f \in R[x]$.

(c) If $c_1 f_1 = c_2 f_2$ *and* $cont(f_1) = cont(f_2) = 1$ *then* f_1, f_2 *are associates in* $R[x]$.

Exercise 3.2. *Let $V = \mathbb{C}^2$, and consider V as a $\mathbb{C}[x]$ module where the multiplication by x is defined to be multiplication by* $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$. *Let $W = \mathbb{C}^2$, and consider W as a $\mathbb{C}[x]$ module where the multiplication by x is defined to be multiplication by* $\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$.

(a) Prove that W, V are $\mathbb{C}[x]$ modules.

(b) Prove that $W \not\simeq V$ as $\mathbb{C}[x]$ modules.

Exercise 3.3. *Consider the following exact sequence.*

$$0 \longrightarrow M' \xrightarrow{\ f\ } M \xrightarrow{\ g\ } M'' \longrightarrow 0$$

(a) Prove that if M', M'' are finitely generated then M is finitely generated.

(b) Does the other direction holds? Prove or give a counter example.

(c) Prove that if M is finitely generated then M'' is finitely generated.

Exercise 3.4. *Let $d \in \mathbb{Z}$ be square free. Show that the integral closure of $\mathbb{Z}$ in $\mathbb{Q}[\sqrt{d}]$ is*

$$\begin{cases} \mathbb{Z}[\frac{\sqrt{d}+1}{2}] & d \equiv 1 \mod 4 \\ \mathbb{Z}[\sqrt{d}] & otherwise \end{cases}$$

Exercise 3.5. *Let F be a field. A linear subspace C of F^n is said to be* cyclic *if whenever $(c_0, c_1, \ldots, c_{n-1}) \in C$ it follows that $(c_{n-1}, c_0, \ldots, c_{n-2}) \in C$.*

(a) Prove that there is a bijection between the monic divisors of $x^n - 1$ in $F[x]$ and the ideals of $F[x]/\langle x^n - 1\rangle$.

(b) Prove that there is a bijection between the ideals of $F[x]/\langle x^n - 1\rangle$ and the cyclic subspaces of F^n.

(c) Find all cyclic subspaces of $\mathbb{F}_2^7$ with dimension 4. Hint: $(x+1)(x^3+x+1)(x^3+x^2+1)$ is the factorization of $x^7 - 1$ to irreducible factors over $\mathbb{F}_2$.