AG codes - Spring 2022
Problem Set 02

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Problem 1. *Let φ, φ' be places of a field F. Show that φ and φ' are equivalent $\iff$ that is a function $\lambda : \varphi(F) \to \varphi'(F)$, with $\lambda(\infty) = \infty$, $\lambda : \varphi(F) \setminus \{\infty\} \to \varphi'(F) \setminus \{\infty\}$ is a field isomorphism and $\varphi' = \lambda \circ \varphi$.*

Problem 2. *Prove that the p-adic valuations (for prime p) are all the non trivial valuations of $\mathbb{Q}$.*

Problem 3. *Finish the claim we saw in class: Let F/F_0 be an algebraic extension, and let $\varphi : F_0 \to L \cup \{\infty\}$ be a place (where L is algebraically closed), with a corresponding valuation ring R_0. We saw in class how to construct a valuation ring $R_0 \subset R \subset F$, and $\tilde{\varphi} : R \to L$ such that $\tilde{\varphi}|_{R_0} = \varphi|_{R_0}$. Prove that we can construct a place $\varphi' : F \to L \cup \{\infty\}$ such that $\varphi(x) = \varphi'(x)$ for all $x \in F_0$.*

Problem 4. *Let $\mathfrak{a}$ be a divisor.*

(a) *Assume* $\deg \mathfrak{a} = 0$. *Prove that the following conditions are equivalent:*

- *(i)* $\mathfrak{a}$ *is principle.*
- *(ii)* $\dim \mathfrak{a} \geq 1$.
- *(iii)* $\dim \mathfrak{a} = 1$.

(b) *Conclude that if* $\deg \mathfrak{a} = 0$ *and* $\mathfrak{a}$ *is not principal then* $\dim \mathfrak{a} = 0$*, and that if* $\mathfrak{a}$ *is principal then* $\dim \mathfrak{a} = 1$ *and* $\deg \mathfrak{a} = 0$.

Problem 5. *Let* $\mathfrak{a} \in \mathcal{D}$ *with* $\deg \mathfrak{a} \geq 0$. *Show that* $\dim \mathfrak{a} \leq \deg \mathfrak{a} + 1$.

Problem 6. *Prove that for every* $\mathfrak{a} \in \mathcal{D}, \mathfrak{a} \geq 0$, *and* $1 \leq k \in \mathbb{N}$,

$$\dim((k-1)\mathfrak{a}) \leq dim(k\mathfrak{a}) \leq \dim((k-1)\mathfrak{a}) + \deg \mathfrak{a}.$$

Problem 7. *Let K be an infinite field and let E/K be a function field.*

(a) *Let* $\mathfrak{a}, \mathfrak{b} \geq 0$, *show that* $\dim \mathfrak{a} + \dim \mathfrak{b} \leq 1 + \dim(\mathfrak{a} + \mathfrak{b})$.

Hint 1: Assume that $\dim(\mathfrak{a} - \mathfrak{p}) < \dim \mathfrak{a}$ *for all* $\mathfrak{p} \in \mathcal{P}$, *then try to eliminate this assumption.*

Hint 2: Under the assumptions of the previous hint, Show that $T := \mathcal{L}(a) \setminus \bigcup_{\mathfrak{p} \in supp\ \mathfrak{b}} \mathcal{L}(a - \mathfrak{p}) \neq \{0\}$, *and consider for* $z \in T$ *the following map from* $\mathcal{L}(\mathfrak{b}) \to \mathcal{L}(\mathfrak{a} + \mathfrak{b})/\mathcal{L}(\mathfrak{a})$, $x \to [z \cdot x] \mod \mathcal{L}(\mathfrak{a})$. *What is the kernel of this map?*

(b) *Conclude that the same holds for any* $\mathfrak{a}, \mathfrak{b}$ *with* $\deg \mathfrak{a}, \deg \mathfrak{b} \geq 0$.