Algebraic Geometry for Theoretical Computer Science

Assignment 6

Lecturer: Gil Cohen *Hand in date: December 11, 2014*

Instructions: Please write your solutions in LaTeX / Word or exquisite handwriting. Submission can be done individually or in pairs.

In this exercise we will study the *field of constants*, defined as follows.

Definition 1 *Let F/K be a function field. The* field of constants *of F/K is defined by*

$$\tilde{K} = \{z \in F \mid z \text{ is algebraic over } K\}.$$

Why care about the field of constants? The motivation for this definition is the following. It could be the case that in the field extension $F/\mathbb{F}_q$, where $\mathbb{F}_q$ is the field of q elements, the elements of $\mathbb{F}_{q^2}$, say, are contained in F. Thus, even though we "started" from $\mathbb{F}_q$, we got the elements of $\mathbb{F}_{q^2}$ inside F.

1. Give an example of an algebraic function field $F/\mathbb{F}_2$ such that $\mathbb{F}_4 \subseteq F$.

Let F/K be a field extension. Recall that the extension is called *finite* if $[F : K] < \infty$. The extension F/K is called *algebraic* if any $x \in F$ is algebraic over K. That is, there exists a polynomial f with coefficients in K, such that $f(x) = 0$.

2. Prove that any finite extension is algebraic.

3. For an element $a \in F$, consider the field $K(a)$ obtained by adjoining a to K. Prove that $K(a)/K$ is a finite extension.

We are now ready to prove that $\tilde{K}$ is a field. This is not obvious – if a, b are algebraic, namely, there exist f_a, f_b polynomials over K, such that $f_a(a) = f_b(b) = 0$, what should be the polynomial over K having root $a + b$?

4. Prove that $\tilde{K}$ is a field. *Guidance: given $a, b \in \tilde{K}$, use the previous two items to show that $K(a, b)$ is an algebraic extension of K.*

Informally speaking, in the rest of the exercise you will be asked to show that K can be "replaced" by $\tilde{K}$ in the results we have seen so far during the course. From here on, F/K is an algebraic function field, $\mathcal{O}$ is a valuation ring of F/K, with the corresponding place P, and discrete valuation v.

5. Show that $\tilde{K} \subseteq \mathcal{O}$, and that $\tilde{K} \cap P = \{0\}$.

6. Let $x \in \tilde{K}$. Prove that $v(x) = 0$.

7. In class we showed that K is embedded in F_P, the residue class field of P. Extend this and show that $\tilde{K}$ is embedded in F_P.
8. Why does $\tilde{K}$ called the field of constants of F/K?