Introduction to Algebraic-Geometric Codes **Spring 2019**

Exercise 2

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Exercise 2.1. *Let R be a UFD, prove that every irreducible element in R is prime.*

Exercise 2.2. *Let R be a ring. Show that the following two conditions are equivalent:*

(i) Every ideal in R can be generated by finitely many elements.

(ii) R satisfies the ascending chain condition: every (infinite) ascending chain of ideals $I_1 \subset I_2 \subset I_3 \subset \dots$ is stationary, i. e. we must have $I_m = I_{m+1} = I_{m+2} = \dots$ for some m.

If R satisfies these conditions, it is called Noetherian

Definition. *Let R be an integral domain. A Euclidean norm in R is a map $N : R \setminus 0 \to \mathbb{N}$ satisfying the following.*

For every $a, b \in R$ with $b \neq 0$, there exist $q, r \in R$ such that:

- $a = qb + r$
- $N(r) < N(b)$ *or* $r = 0$

A Euclidean domain is an integral domain with a Euclidean norm.

Exercise 2.3. *Prove that every Euclidean domain is a PID.*

Exercise 2.4. *Let F be a field, prove that $F[x]$ is a Euclidean domain.*

Exercise 2.5. *Let $p \in \mathbb{Z}$ be a prime number, and let $\overline{g}(x) \in (\mathbb{Z}/p\mathbb{Z})[x]$ be any irreducible polynomial. Let $g(x) \in \mathbb{Z}[x]$ be such that it image under the natural reduction $(\mathbb{Z}/p\mathbb{Z})[x]$ is $\overline{g}(x)$. Show that the ideal $\langle p, g(x) \rangle$ is maximal in $\mathbb{Z}[x]$*