Free Probability and Ramanujan Graphs - HW #2

Published: 2024-06-18 *Submission: 2024-07-02*

Exercises with * will be graded. Submission in singles or pairs. Contact Gal for questions and clarifications.

1. * In this question we consider the $*$-probability space $(\mathbb{C}G, \tau_G)$ and the element

$$\Delta = g + h + g^{-1} + h^{-1},$$

where $g, h \in G$ are distinct elements of infinite order, and none of them generates G by itself.

   (a) Consider the (infinite) Cayley graph whose vertices are the elements of G, and the set of generators are $\{g, h, g^{-1}, h^{-1}\}$. Prove that for any $n \geq 1$, the moment $\tau_G(\Delta^n)$ equals the number of closed walks of length n in the graph, originating at an arbitrary vertex (and in particular the identity element e_G).

   (b) For the additive group $G = \mathbb{Z}^2$ let $g = (1, 0)$ and $h = (0, 1)$. Prove that

$$\tau_G(\Delta^n) = \begin{cases} \binom{2p}{p}^2, & n = 2p, \\ 0, & n \text{ is odd.} \end{cases}$$

   (**Hint**: consider the set UR of steps going either up or right, and the set DR of steps going either down on right. Notice that a walk of length n ending at (a, b) satisfies $a + b = |UR| - (n - |UR|)$).

   (c) Let G be the *free group on two generators*, with g and h being the generators. Calculate $\tau_G(\Delta^n)$ for $n = \{0, 1, 2, 3, 4\}$.

   (d) Provide lower and upper bounds for $\tau_G(\Delta^3)$ and $\tau_G(\Delta^4)$ for any group G under the given assumptions.

2. In this question we will prove that $C_n = \frac{1}{n+1}\binom{2n}{n}$.

   (a) Prove that the number of NE-SE paths which end at a given point $(a, b) \in \mathbb{Z}^2$ is $\binom{m}{\frac{m+n}{2}}$, where $m \geq |n| \geq 0$ and $m + n$ is even (and 0 otherwise). Observe that in particular, $\binom{2p}{p}$ such paths are ending at $(2p, 0)$.

   (b) Prove that there is a bijection between *bad* paths, ending at $(2p, 0)$ but are not Dyck paths, and paths of length $2p$ ending at $(2p, -2)$. (**Hint**: Let γ be a bad path, and let j be the first step in which γ went below height 0. Define the bijection to be the reflection of the path to the right of step j across the line -1).

   (c) Conclude that $C_n = \binom{2n}{n} - \binom{2n}{n-1}$.

3. * Express the following counting problems using the Fuss-Catalan numbers $C_k^{(p)}$, and explain your answer.

(a) How many non-crossing pairings are there for the numbers $[2n] = \{1, 2, \ldots, 2n\}$? (A pairing is a partition $V_1, V_2, \ldots, V_n$ such that $\bigcup_{i=1}^{n} V_i = [2n]$ and $|V_i| = 2$ for every i. A partition is said to be *crossing* if there exist $V_i = (a, b)$ and $V_j = (c, d)$ such that $a < c < b < d$).

(b) How many ways are there to partition the numbers $[2n]$ to non-crossing subsets of even size?

(c) *Knights and ladies of the Round Table.* There are $k = ab$ ladies, each one accompanied by her knight. How many ways are there to sit them together such that the ladies and knights are in groups of size a (that is, there are a ladies sitting together, next to them a knights, next to them a ladies, and so on) and each lady can converse with her knight with no two conversations crossing.